Jurnal Tadris Matematika (JTMT)
Jurnal of Tadris Matematika (JTMT)
Volume 5, Issue 2, 2024
E-ISSN : 2745-9551
P-ISSN : 2745-956X
https://journal.uiad.ac.id/index.php/Jtm

Formulas For Calculating The Areas Of Regiones Bounded By The Quadratic Function And The Line, Two Quadratic Functions

Abdiyeva Shoira Shukurovna
PhD student of Chirchik State pedagogical university, Uzbekistan
E-mail correspondence: abdiyevashoira08@gmail.com
DOI: 10.47435/jtmt.v5i2.3272

Submission Track:
||Accepted: November 3, 2024.||Approved: December 1, 2024||Published: December 30, 2024

Abstract

This article brings formulas for finding the areas of regiones bounded by the quadratic function and the x-axis, the quadratic function and the line $y = kx + l$ *, two quadratic functions and these three cases are generalized. The generated formula can be used in problems concerning the calculation of the area of the region bounded by the quadratic function. In the secondary schools mathematics course, the define integral topic presents problems of the quadratic function concerning with the area. The article is viewed for the general equation of the quadratic function and induced a formula related to the discriminant, roots and coefficient of* x^2 *term of the quadratic polinomial. Vieta's theorem was used to create formulas. From the geometric point of view, the area bounded by the x-axis and the quadratic function equal to* $\frac{2}{3}$ *part of the rectangle, which sides equal to the subtraction of quadratic polinomial roots and the ordinate of vertex of a quadratic function (parabola). In conclusion two formulas are induced. Second formula is related to the ordinate of vertex of a quadratic function (parabola). These formulas allow to solve the problems under consideration quickly and easily.*

Keywords: area of the region; coefficients; discriminant; formula; quadratic function; roots.

1. Introduction

In the secondary schools of mathematics course (Linsky, J., Western, B., & Nicholson, J., 2018) is provides examples of calculating the areas of the regions bounded by the quadratic function and the *x*-axis, the quadratic function and the line $y = kx + l$ and two quadratic functions in examples relating to define integral. Most countries mathematics books (Муравина, О., & Муравин, Г., 2020) of secondary school exist topics about an application of definite integral. Secondary school often experience difficulty with solving integration problems. Integration is part of the Mathematics syllabus in most of countries (Kiat, 2005; Лукьянова, Т. И., & Мансурова, Е. Р., 2017; Danial, 2020).

Archimed learned a parabola as the section produced by a plane cutting a right angled cone at a right angle to an element of the cone (Bergsten, 2004). One of Archimedes' most celebrated discoveries is his determination of the area of a segment of a parabola (Dijksterhuis, E.J. translated by C.Dikshoorn, 1987). Of course, the area of a parabolic segment can be found today with integral calculus. Isaac Newton and Gottfried Wilhelm Leibniz, who formally developed calculus (Bardi, 2009). Archimedes' approach to area was foundational in the eventual formulation of the integral (Powers, 2020; Lu, 2023).

Problems of the quadratic function and quadratic equation have been studied in the work of researchers (Díaz, V., Aravena, M., & Flores, G., 2020; Shinariko, L. J., Hartono, Y., Yusup, M., Hiltrimartin, C., & Araiku, J., 2021; Tendere, J., & Mutambara, L. H. N., 2020).

Geometrical and graphical solution of quadratic equations (Hornsby, E. J., 1990; Ben-Ari, M., 2022) and history of quadratic equation (Taub, D., 2022; Rogers, L. & Pope, S., 2015) learned by researchers.

The Vieta's theorem is used for relating coefficients to roots of quadratic polynomial (Абдієва, Ш., & Тургунбаєв, Р., 2023).The generalized form of the Vieta's Theorem and its special cases for quadratic, cubic, and quartic equations considered by Grigorieva, E., & Grigorieva, E. (2015).

Researchers (Hidayah et al., 2021;Susilo, B. E., 2019; Camacho, M., 2004; Serhan, D., 2015; Ely, R., & Jones, S. R., 2023) have applied the define integral to solving problems related to finding the area of the region.

But the problems which are related coefficients of quadratic polynomial to the areas of the regions bounded by the quadratic function and the *x*-axis, the quadratic function and the line $y = kx + l$ and two quadratic functions have not considered.

How to solve finding the areas of the regions bounded by the quadratic function and the *x*-axis, the quadratic function and the line $y = kx + l$ and two quadratic functions without applying the define integral?

How is related coefficients of quadratic polinomial to areas of the regiones bounded by the quadratic function and the *x*-axis, the quadratic function and the line $y = kx + l$ and two quadratic functions? How is represented found region by geometrical point of view? It has been established below that solutions to such problems can be expressed in a general formula. Below formula created for three cases and they generalized.

2. Method

Formula for finding the area of region bounded by the quadratic function and the x-axis

If diskriminant of $ax^2 + bx + c = 0$ quadratic polinomial is $D > 0$ positive, then it has $x_1, x_2 \left(x_1 < x_2\right)$ real roots exist. How is related the coefficients of quadratic polynomial to the area bounded by the $y = ax^2 + bx + c$ quadratic function and the *x*-axis? Below it was calculated the define integral of the quadratic function $y = ax^2 + bx + c$ on the interval $[x_1; x_2]$ (figures 1-2).

Since Vieta's theorem $x_1 + x_2 = -\frac{b}{a}$, $x_1 \cdot x_2 = \frac{c}{a}$ is appropriate for the quadratic equation $ax^2 + bx + c = 0$, it will be calculate following define integral:

$$\int_{x_1}^{x_2} (ax^2 + bx + c)dx = \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\Bigg|_{x_1}^{x_2} = \frac{a}{3}\left(x_2^3 - x_1^3\right) + \frac{b}{2}\left(x_2^2 - x_1^2\right) + c\left(x_2 - x_1\right) =$$

$$= \frac{a}{3}\left(x_2 - x_1\right)\left(x_2^2 + x_2x_1 + x_1^2\right) + \frac{b}{2}\left(x_2^2 - x_1^2\right) + c\left(x_2 - x_1\right) = \left(x_2 - x_1\right)\left(\frac{a}{3}\left(\left(x_1 + x_2\right)^2 - x_1x_2\right) + \frac{b}{2}\left(x_1 + x_2\right) + c\right) =$$

$$= \left(x_2 - x_1\right)\left(\frac{a}{3}\left(\left(-\frac{b}{a}\right)^2 - \frac{c}{a}\right) + \frac{b}{2}\left(-\frac{b}{a}\right) + c\right) = \left(x_2 - x_1\right)\left(\frac{b^2}{3a} - \frac{c}{3} - \frac{b^2}{2a} + c\right) = \left(x_2 - x_1\right)\frac{-\left(b^2 - 4ac\right)}{6a} =$$

$$= \left(x_2 - x_1\right)\frac{-D}{6a}.$$

From Figure 1 it can be seen that the graph of a function lies below the *x*-axis on the interval $[x_1; x_2]$ i.e. the value of the function is negative. However, when the area of the region is calculated, its absolute value is obtained.

If $a > 0$, then it will be $\frac{-D}{6a} < 0$, but since the area of the region is positive, it will be get its absolute value.

In conclusion, it can be define that the area of bounded by $y = ax^2 + bx + c$ quadratic function and the *x*-axis depends on the value of discriminant *D* and coefficient *a*:

$$S = \left| (x_2 - x_1) \frac{D}{6a} \right|, \quad (1)$$

It is appropriate when the formula (1) induced above is $D > 0$, that is, a quadratic function has two common points with the *x*-axis.

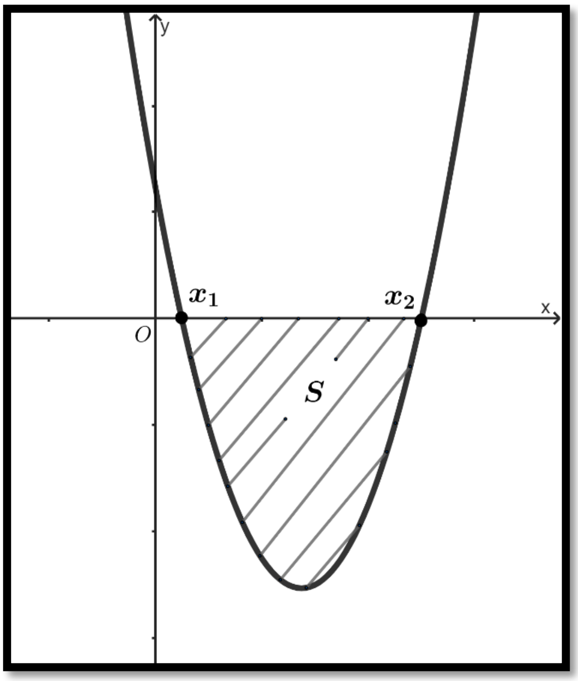

Figure 1. Area of region bounded by the quadratic function and the *x*-axis

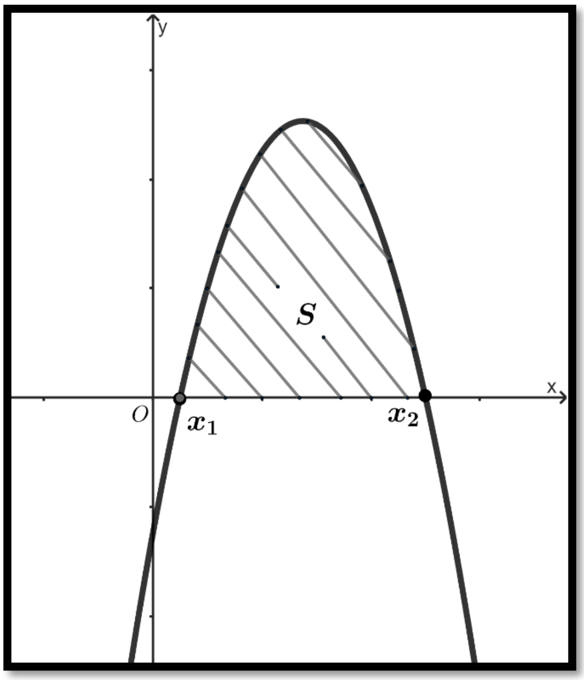

Figure 2. Area of region bounded by the quadratic function and the *x*-axis

Formula for finding the area of region bounded by the quadratic function and the line $y = kx + l$

Now, when a straight line is taken in place of the *x*-axis, that is, the question of how to find the area by the curve with equation $y = ax^2 + bx + c$ and the line $y = kx + l$ is considered (Figure 3-4).

When intersecting with $y = kx + l$ line and $y = ax^2 + bx + c$ parabola, the quadratic equation $-ax^2 - (b-k)x - (c-l) = 0$ is obtained. For the resulting quadratic equation must be $D > 0$, otherwise the parabola and line will not intersect and an enclosed region will not be formed. If $D > 0$, then the equation will have x_1 and x_2 roots.

Below, when $a > 0$, the area is found for the region bounded by the quadratic function $y = ax^2 + bx + c$ (parabola) and the line $y = kx + l$ (Figure 3).

Jurnal Tadris Matematika (JTMT)
Jurnal of Tadris Matematika (JTMT)
Volume 5, Issue 2, 2024
E-ISSN : 2745-9551
P-ISSN : 2745-956X
https://journal.uiad.ac.id/index.php/Jtm

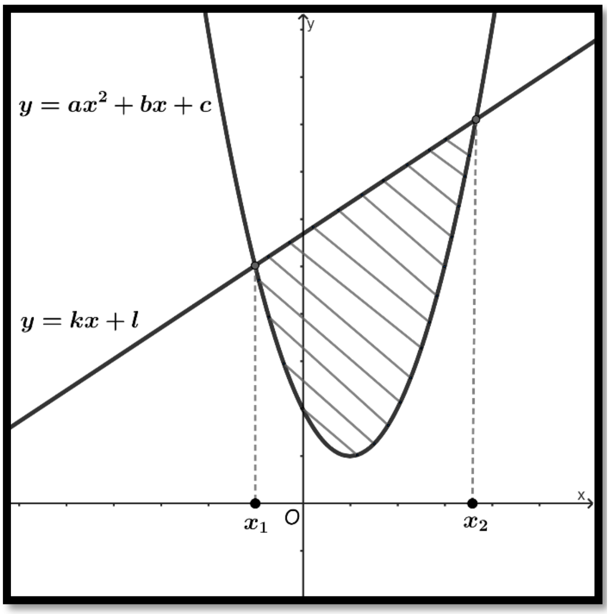

Figure 3. Area of region bounded by the quadratic function and the line $y = kx + l$

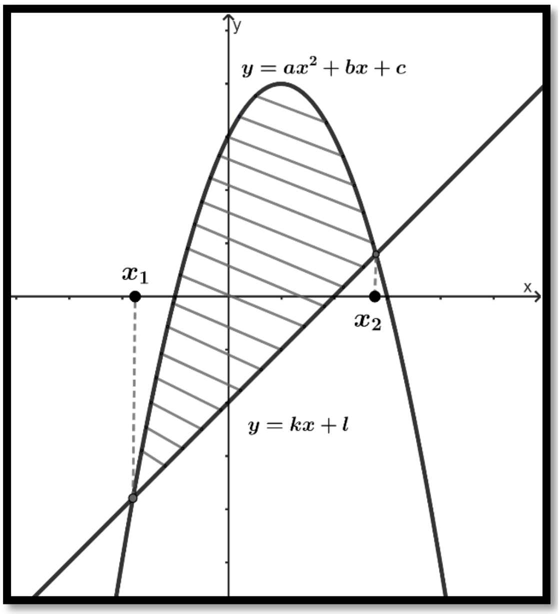

Figure 4. Area of region bounded by the quadratic function and the line $y = kx + l$

$$S = \int_{x_1}^{x_2} \left((kx + l) - (ax^2 + bx + c) \right) dx = \int_{x_1}^{x_2} \left(-ax^2 - (b-k)x - (c-l) \right) dx =$$

For the quadratic polinomial $-ax^2 - (b-k)x - (c-l)$ under the integral, the Vieta's theorem is appropriate and becomes $x_1 + x_2 = -\frac{b-k}{a}$, $x_1 x_2 = \frac{c-l}{a}$. Then

$$= \left(-\frac{ax^3}{3} - \frac{(b-k)x^2}{2} - (c-l)x \right) \Bigg|_{x_1}^{x_2} = -\frac{a}{3}\left(x_2^3 - x_1^3\right) - \frac{b-k}{2}\left(x_2^2 - x_1^2\right) - (c-l)(x_2 - x_1) =$$

$$= -(x_2 - x_1)\left(\frac{a}{3}\left((x_1 + x_2)^2 - x_1 x_2 \right) + \frac{b-k}{2}(x_1 + x_2) + (c-l) \right) =$$

$$= -(x_2 - x_1)\left(\frac{a}{3}\left(\left(-\frac{b-k}{a}\right)^2 - \frac{c-l}{a} \right) + \frac{b-k}{2}\left(-\frac{b-k}{a}\right) + (c-l) \right) =$$

$$= -(x_2 - x_1)\left(\frac{(b-k)^2}{3a} - \frac{c-l}{3} - \frac{(b-k)^2}{2a} + (c-l) \right) =$$

$$= (x_2 - x_1)\frac{(b-k)^2 - 4a(c-l)}{6a} = (x_2 - x_1)\frac{D}{6a}.$$

So, when $a > 0$, the area bounded by the parabola and the line is calculated by the formula

$$S = (x_2 - x_1)\frac{(b-k)^2 - 4a(c-l)}{6a} = (x_2 - x_1)\frac{D}{6a} \qquad (2)$$

Formula for finding the area of region bounded by two quadratic functions

Below is the problem of finding the area bounded by two curves $y = ax^2 + bx + c$ and $y = mx^2 + nx + p$ (Figures 5-6).

Jurnal Tadris Matematika (JTMT)
Jurnal of Tadris Matematika (JTMT)
Volume 5, Issue 2, 2024
E-ISSN : 2745-9551
P-ISSN : 2745-956X
https://journal.uiad.ac.id/index.php/Jtm

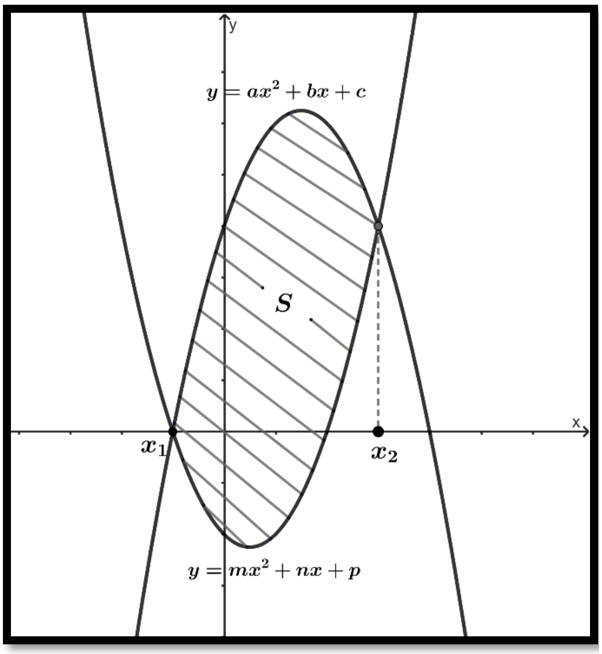

Figure 5. The area of region bounded by two quadratic functions

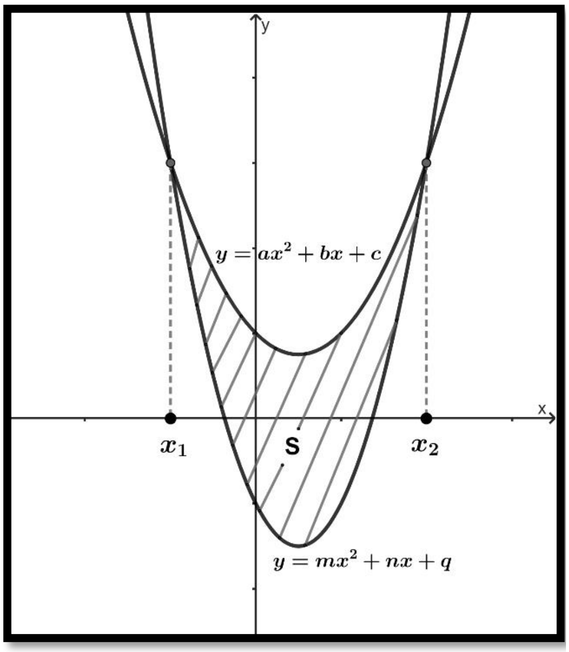

Figure 6. The area of region bounded by two quadratic functions

$$S = \int_{x_1}^{x_2}\left(\left(ax^2+bx+c\right)-\left(mx^2+nx+p\right)\right)dx = \int_{x_1}^{x_2}\left(\left(a-m\right)x^2+\left(b-n\right)x+\left(c-p\right)\right)dx =$$

$$= \left(-\frac{ax^3}{3}-\frac{\left(b-k\right)x^2}{2}-\left(c-l\right)x\right)\Bigg|_{x_1}^{x_2} = \frac{a-m}{3}\left(x_2^3-x_1^3\right)+\frac{b-n}{2}\left(x_2^2-x_1^2\right)+\left(c-p\right)\left(x_2-x_1\right)=$$

$$=\left(x_2-x_1\right)\left(\frac{a-m}{3}\left(\left(x_1+x_2\right)^2-x_1x_2\right)+\frac{b-n}{2}\left(x_1+x_2\right)+\left(c-p\right)\right)=$$

$$=\left(x_2-x_1\right)\left(\frac{a-m}{3}\left(\left(-\frac{b-n}{a-m}\right)^2-\frac{c-p}{a-m}\right)+\frac{b-n}{2}\left(-\frac{b-n}{a-m}\right)+\left(c-p\right)\right)=$$

$$=\left(x_2-x_1\right)\left(\frac{\left(b-n\right)^2}{3\left(a-m\right)}-\frac{c-p}{3}-\frac{\left(b-n\right)^2}{2\left(a-m\right)}+\left(c-p\right)\right)=$$

$$=\left(x_2-x_1\right)\frac{-\left(\left(b-n\right)^2-4\left(a-m\right)\left(c-p\right)\right)}{6\left(a-m\right)}=\left(x_2-x_1\right)\frac{-D}{6\left(a-m\right)}$$

Since $a-m<0$, $\frac{-D}{6\left(a-m\right)}>0$. Here, the fact that Vieta's theorem $x_1+x_2=-\frac{b-n}{a-m}$, $x_1x_2=\frac{c-p}{a-m}$ has been used for the quadratic equation $\left(a-m\right)x^2+\left(b-n\right)x+\left(c-p\right)=0$ under the integral. $y=ax^2+bx+c$ and $y=mx^2+nx+p$ quadratic functions must intersect at two points, otherwise a bounded enclosed region will not be formed. Where $a-m\neq 0$ is the coefficient of x^2 term of the quadratic polinomial $\left(a-m\right)x^2+\left(b-n\right)x+\left(c-p\right)$ under the integral and discriminant $D>0$ is positive.

So, area bounded by two curves with equations $y=ax^2+bx+c$ and $y=mx^2+nx+p$ calculated by following formula:

$$S=(x_2-x_1)\frac{-\left((b-n)^2-4(a-m)(c-p)\right)}{6(a-m)}=(x_2-x_1)\frac{-D}{6(a-m)} \qquad (3)$$

3. Results and Discussion

Generalized formula for above three cases

In conclusion, all three cases of the above, the area of the region is represented by the similar formulas:

$$S=\left|(x_2-x_1)\frac{D}{6q}\right|, \qquad (4)$$

where x_1, x_2 is the abssissas of intersection points of graphs, D is the discriminant of the quadratic polinomial under the integral, q is the coefficient of x^2 term of the quadratic polinomial under the integral.

Geometric point of view.

The founding formula (1) is calculated the area of the region bounded by the *x*-axis and the quadratic function. Below, from the fact that this region lies inside rectangle which sides equal to x_2-x_1 and y_0, the finding region forms how part of the area of the rectangle (Figure 7).

$x_0=\frac{x_1+x_2}{2}=-\frac{b}{2a}$ is abssissa of vertex of parabola, $y_0=y(x_0)=ax_0^2+bx_0+c=a\cdot\left(-\frac{b}{2a}\right)^2+b\cdot\left(-\frac{b}{2a}\right)+c=\frac{-(b^2-4ac)}{4a}=\frac{-D}{4a}$ is ordinate of vertex of parabola. Since the ordinate of $y=ax^2+bx+c$ parabola is $y_0=\frac{D}{4a}$, it is expressed by the formula (1) found above.

$$S=\left|(x_2-x_1)\frac{D}{6a}\right|=\left|(x_2-x_1)\cdot\frac{2}{3}\cdot\frac{D}{4a}\right|=\left|(x_2-x_1)\frac{2}{3}y_0\right| \qquad (5)$$

If the (5) formula is viewed from a geometric point of view, the area of the region bounded by the *x*-axis and the quadratic function is equal to $\frac{2}{3}$ of the area of the rectangle which sides equal to x_2-x_1 and y_0. The area of the shaded region is $\frac{2}{3}$ part of the area of the rectangle. The remainder part is equal to $\frac{1}{3}$ part of the rectangle. So when you calculate the area of a region, you have to take into account the vertex of the parabola (Figure 7).

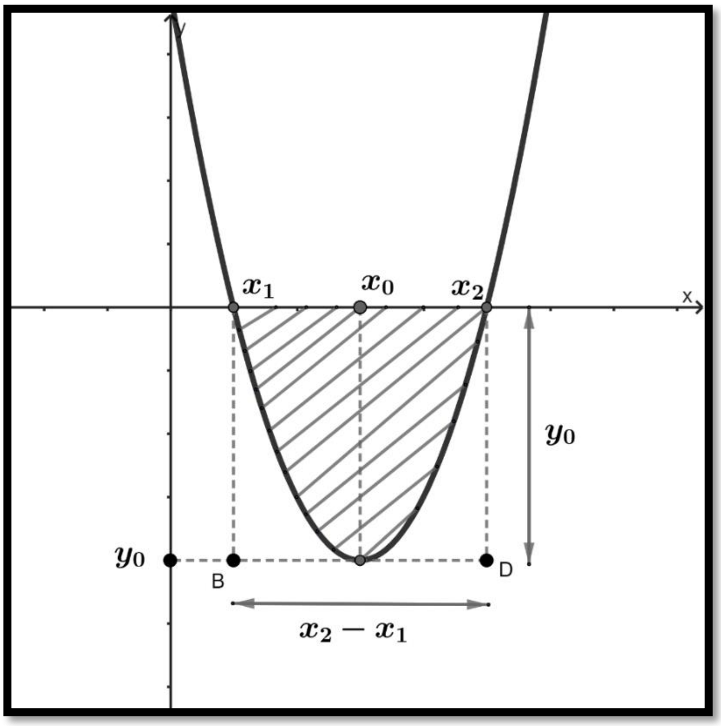

Figure 7

4. Conclusion

So, since the area of the region presented in Figure 7 is also expressed using the following formula:

$$S = \left|\left(x_2 - x_1\right)\frac{2}{3}y_0\right| \quad (6)$$

where x_1, x_2 are abssissas of points of intersection with the *x*-axis of a quadratic function graph, y_0 is the ordinate of vertex of the parabola.

Second formula for above three cases

For all of the above cases, the formula (7) expressed through the ordinate of vertex of the parabola is appropriate.

$$S = \left|\left(x_2 - x_1\right)\frac{D}{6q}\right| = \left|\left(x_2 - x_1\right)\frac{2}{3}y_0\right|, \quad (7)$$

where y_0 is the ordinate of vertex of a quadratic function (parabola) under the integral.

So it can solve that the above problems can be solved using two formulas (4) and (6). Also, these formulas allow you to quickly and easily solve such problems.

Bibliography

Bardi, J. S. (2009). *The calculus wars: Newton, Leibniz, and the greatest mathematical clash of all time.* Hachette UK.

Ben-Ari, M. (2022). *Solving Quadratic Equations. In: Mathematical Surprises.* Springer, Cham. doi:10.1007/978-3-031-13566-8_7

Bergsten, C. (2004). Beyond the representation given: The parabola and historical metamorphoses of meanings. (10), 37-49.

Camacho, M., Depool, R., & Santos-Trigo, L. M. (2004). Promoting Students Comprehension of Definite Integral and Area Concepts Through the Use of Derive Software". Proceedings of the 26 PME-NA, 2, 447-454. *Proceedings of the 26 PME-NA*, (стр. 447-454).

Danial, D. (2020). KETERLAKSANAAN KURIKULUM 2013 DALAM MATA PELAJARAN MATEMATIKA DI SMP NEGERI 33 MAKASSAR. *JTMT: Journal Tadris Matematika, 1*(1), 27-32. doi:10.47435/jtm.v1i1.395

Díaz, V., Aravena, M., & Flores, G. (2020). Solving problem types contextualized to the quadratic function and error analysis: A case study. *EURASIA Journal of Mathematics, Science and Technology Education, 16*(11). doi:10.29333/ejmste/8547

Dijksterhuis, E.J. translated by C.Dikshoorn. (1987). *Archimedes.* Princeton University.

Ely, R., & Jones, S. R. (2023). The teaching and learning of definite integrals: A special issue guest editorial. *International Journal of Research in Undergraduate Mathematics Education, 9*(1), 1-7. doi:10.1007/s40753-023-00214-2

Grigorieva, E. &. (2015). *Polynomials. In: Methods of Solving Nonstandard Problems.* Birkhäuser, Cham. doi:10.1007/978-3-319-19887-3_2

Hidayah, N., Danial, D., & Takdir, T. (2021). Diagnostik Kesulitan Belajar Mahasiswa Pada Mata Kuliah Kalkulus Program Studi Tadris Matematika IAIM Sinjai. *JTMT: Journal Tadris Matematika, 2*(2), 31-39. doi:10.47435/jtmt.v2i2.728

Hornsby, E. J. (1990). Geometrical and Graphical Solutions of Quadratic Equations. *The College Mathematics Journal, 21*(5), 362-378. doi:10.1080/07468342.1990.11973334

Kiat, S. E. (2005). Analysis of Students' Difficulties in Solving Integration Problems. *The Mathematics Educator, 9*(1), 39-59. Получено из fliphtml5.com/aqlj/sogn/basic

Linsky, J., Western, B., & Nicholson, J. (2018). *Complete Pure Mathematics 1 for Cambridge International AS & A level.* Oxford University Press - Children.

Lu, J. (2023). Who Contributed More on Calculus? Newton or Leibniz? *Studies in Social Science & Humanities, 2*(7), 22-25. doi:10.56397/sssh.2023.07.04

Parwati, G., Hadjar, I., Pathuddin, P., & Alfisyahra, A. (2023). Analysis of the Error Students in Solving Story Problems on Quadratic Equation Material Based on Newman Stages. *JTMT: Journal Tadris Matematika, 4*(02), 208-220. doi:10.47435/jtmt.v4i02.1976

Powers, J. (2020). *Did Archimedes Do Calculus?*

Rogers, L., & Pope, S. (2015). A brief history of quadratic equations for mathematics educators. *Proceedings of the British Society for Research into Learning Mathematics, 35*(3), 90-95.

Serhan, D. (2015). Students' understanding of the definite integral concept. International Journal of Research in Education and Science (IJRES). *1*(1), 84-88.

Shinariko, L. J., Hartono, Y., Yusup, M., Hiltrimartin, C., & Araiku, J. (2021). Mathematical Representation Ability on Quadratic Function Through Proof Based Learning. *In 4th Sriwijaya University Learning and Education International Conference (SULE-IC 2020)* (стр. 653-659). Atlantis Press.

Susilo, B. E., Darhim, D., & Prabawanto, S. (2019). Students critical thinking skills toward concepts differences in finding area of a plane region and definite integral. Unnes Journal of Mathematics

Education, 8(1), 1-7. https://journal.unnes.ac.id/sju/i. *Unnes Journal of Mathematics Education, 8*(1), 1-7. Получено из https://journal.unnes.ac.id/sju/i

Taub, D. (2022). *A Brief History of Quadratic Equations.* Universitetstryckerie.

Tendere, J., & Mutambara, L. H. N. (2020). An analysis of errors and misconceptions in the study of quadratic equations. *European Journal of Mathematics and Science Education, 1*(2), 81-90. doi:10.12973/ejmse.1.2.81

Абдієва, Ш., & Тургунбаєв, Р. (2023). THEOREMS ON THE NUMBER OF ROOTS OF A CUBIC EQUATION AND THEIR LOCATION AS A MEANS OF DEVELOPING STUDENTS' VISUAL THINKING/10.3110/2413-1571-2023-038-4-001. *Фізико-математична освіта, 38*(4), 7-13. doi:10.31110/2413-1571-2023-038-4-001

Лукьянова, Т. И., & Мансурова, Е. Р. (2017). Интеграл в школьном курсе математики. Физико-математические образование: проблемы и перспективы: сб. статей.–Елабуга. *Изд-во Казан. Ун-та*, (стр. 51-55).

Муравина, О., & Муравин, Г. (2020). *Алгебра и начала математического анализа. 11 класс. Базовый уровен.* Litres.